§3.5,3.7 **Name:** ______________________

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Review

2^{ND} Order Non-homogeneous ODEs

- A linear second order non-homogeneous ODE with constant coefficients can be written as
$$ay'' + by' + cy = g(t) \qquad (*)$$
- The corresponding homogeneous ODE is
$$ay'' + by' + cy = 0 \qquad (**)$$
- If $y_1(t)$ and $y_2(t)$ are the solutions of $(**)$ (a.k.a. *homogeneous or complementary solutions*), and $Y(t)$ is any solution of $(*)$(a.k.a. *particular solution*), then the *general solution* of $(*)$ is written as
$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$$
- If Y_1 is a particular solution of $ay'' + by' + cy = g_1(t)$ and Y_2 is a particular solution of $ay'' + by' + cy = g_2(t)$, then $Y_1 + Y_2$ is a solution of $y'' + P(t)y' + q(t)y = g_1(t) + g_2(t)$.

Undetermined Coefficients

- This is a method to systematically guess a particular solution for $(*)$.
- Only works when $g(t)$ is a sin, cos, exponential, polynomial or a combination of these functions.
- Steps:
  - Solve the homogeneous equation to obtain y_1 and y_2.
  - Make a guess for $Y(t)$. Common initial guess:

$g(t)$:	Guess $Y(t)$ as
t	$At + B$
t^2	$At^2 + Bt + C$
t^3	$At^3 + Bt^2 + Ct + D$
e^{at}	Ae^{at}
$\sin(at)$	$A\sin(at) + B\cos(at)$
$\cos(at)$	$A\sin(at) + B\cos(at)$
Product of functions listed above	Product of guesses listed above

- If your guess matches 1 of the homogeneous solutions, then multiply your guess by t.
- If your guess matches 2 of the homogeneous solutions, then multiply your guess by t^2 (this happens when your homogeneous equation has repeated roots).
- Plug in your guess into $(*)$ and solve for all the constants A, B, $C, ...$ etc. to obtain $Y(t)$.
- If the $g(t)$ is the sum of several terms, do the above steps for each of those terms and add the solutions to get the full particular solution.
- Write down the general solution (by adding the homogeneous solution and particular solution).

Practice Problems

1. Example Problem: Find the general solution of

$$y'' - y' + \frac{1}{4}y = e^{t/2}$$

Using the method of undetermined coefficients.

Solution: First we plug in $e^r t$ to solve the homogeneous problem. This gives us the characteristic equation

$$r^2 - r + \frac{1}{4} = 0$$

and the roots are $\frac{1}{2}$ repeated. Thus $y_1 = e^{t/2}$ and $y_2 = te^{t/2}$. Now, since the right hand side is $e^{t/2}$ we guess $Y(t) = Ae^{t/2}$. We see that this is contained in both the homogeneous solutions, thus we will have to multiply our guess by t^2. Our new guess is $At^2e^{t/2}$.
Let's plug this into the ODE:

$$Y' = \frac{A}{2}t^2e^{t/2} + 2Ate^{t/2}$$

$$Y'' = \frac{A}{4}t^2e^{t/2} + Ate^{t/2} + 2Ae^{t/2} + Ate^{t/2} = \frac{A}{4}t^2e^{t/2} + 2Ate^{t/2} + 2Ae^{t/2}$$

$$Y'' - Y' + \frac{1}{4}Y = \left(\frac{A}{4}t^2e^{t/2} + 2Ate^{t/2} + 2Ae^{t/2}\right) - \left(\frac{A}{2}t^2e^{t/2} + 2Ate^{t/2}\right) + \frac{1}{4}\left(At^2e^{t/2}\right) = 2Ae^{t/2}$$

Thus we need to satisfy $2Ae^{t/2} = e^{t/2}$ or $A = \frac{1}{2}$. We can now write down the general solution as

$$y = c_1e^{t/2} + c_2te^{t/2} + \frac{1}{2}t^2e^{t/2}$$

2. For the following non-homogeneous equations, determine a suitable form for the particular solution $Y(t)$ using the method of undetermined coefficients. **You don't have to find the coefficients!**

(a)

$$y'' + 2y' + 5y = t$$

Solution: Roots of the characteristic equation are $-1 \pm 2i$ and the homogeneous solutions are $y_1 = e^{-t}\cos(2t)$ and $y_1 = e^{-t}\sin(2t)$. Since the right hand side is just t, we pick $Y(t) = At + B$.

(b)

$$y'' + 2y' + 5y = 2te^{-2t}\cos(t)$$

Solution: The homogeneous solution is the same as before. The right hand side has three terms multiplied. There's t, e^{-2t} and $\cos t$. Looking at the table, the corresponding choices of particular solution are $A_1t + B_1$, A_2e^{-2t} and $A_3\cos t +$

$B_3 \sin t$. Thus the particular solution should be the product of these three, i.e. $Y(t) = (At+B)e^{-2t}\cos t + (Ct+D)e^{-2t}\sin t$ where we have collected the constants and given them new names. This is a good guess since it doesn't match with the homogeneous solutions.

(c)

$$y'' + 2y' + 5y = 3te^{-t}\cos(2t)$$

Solution: The homogeneous solution is the same as before. Following the same steps in (b) to arrive at $Y(t) = (At+B)e^{-t}\cos(2t) + (Ct+D)e^{-t}\sin(2t)$. However we notice that this guess has terms of the form $e^{-t}\cos(2t)$ and $e^{-t}\sin(2t)$ which match with the homogeneous solutions we computed in (a). This means that we have to multiply our guess by t and we have $Y(t) = (At+B)te^{-t}\cos(2t) + (Ct + D)te^{-t}\sin(2t)$

3. Solve the initial value problem

$$y'' + y' - 2y = 2t \quad y(0) = 0 \quad y'(0) = 1$$

using the method of undetermined coefficients.

Solution: The homogeneous solution is straightforward to compute: $c_1e^{-2t} + c_2e^t$. Now since the right hand side is t, we guess $Y = At + B$. Thus $Y' = A$ and $Y" = 0$. Plugging this into the equation, we get $(A - 2B) - 2At = 2t$. In order for the equation to hold, the coefficients of t and the constant terms have to match thus $2 = -2A$ and $A - 2B = 0$. Solving these two equations, we get $A = -1$ and $B = -1/2$. The general solution to the ODE is thus given by

$$y(t) = c_1e^{-2t} + c_2e^t - t - \frac{1}{2}.$$

Plugging in the initial conditions, we get $c_1 = \frac{-1}{2}$ and $c_2 = 1$.

Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee`