§3.7,3.8 **Name:** ______________________

Fall MATH 2930 21 October 2021

Review

Free and Forced Vibrations

- The following equation represents vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where m is *mass*, γ is the *damping coefficient*, k is the *spring constant* and $F(t)$ is the *forcing*.

- $\omega_0 = \sqrt{k/m}$ is referred to as the *natural frequency*.
- When $F(t) = 0$, i.e. homogeneous, we refer to the vibrations as *free*.
- Additionally, if $\gamma = 0$ we refer to the vibrations as *undamped*.
- The solution to undamped free vibrations is

$$u(t) = R\cos(\omega_0 t - \delta)$$

thus it oscillates with a constant amplitude forever.

- The solution to damped free vibrations when $0 < \gamma^2 < 4km$ (small damping) is

$$u(t) = R_0 e^{-\gamma t/(2m)} \cos(\omega_0 t - \delta_0)$$

thus the amplitude of oscillation decreases with time due to the exponential factor.

- The forcing, $F(t)$ is usually a sine or cosine when it is present, i.e., $F(t) = F_0 \cos(\omega t)$ where ω is called the *forcing frequency*.
- If $\omega \neq \omega_0$, we have the general solution

$$u(t) = \underbrace{R_0 e^{-\gamma t/(2m)} \cos(\omega_0 t - \delta_0)}_{\text{homogeneous sol.}} + \underbrace{R\cos(\omega t - \delta)}_{\text{particular sol.}}$$

- If $\gamma = 0$ and $\omega = \omega_0$, we have *resonance* and

$$u(t) = R_0 \cos(\omega_0 t - \delta_0) + R\, t \cos(\omega t - \delta)$$

(notice the factor of t in the second term). In this case the amplitude goes to ∞ as t increases.

Practice Problems

1. Example problem: Consider the damped, forced oscillator described by

$$u'' + \lambda u' + u = F_0 \sin(\omega t)$$

where $\lambda > 0$.

(a) Find the steady state (i.e. particular) solution of this equation.

Solution: First, suppose that $\lambda > 0$ (that is λ is not zero). Then we know that the homogeneous solution is going to have term of the form $e^{-\Box t}\sin(\Box t)$ and $e^{-\Box t}\cos(\Box t)$ and since the forcing term (inhomogeneity) is simply a sine (no exponential term) we do not have to worry about multiplying our particular solution by t. Thus we may write

$$Y(t) = A\cos\omega t + B\sin\omega t.$$

Computing the derivatives, we have

$$Y' = -\omega A\sin\omega t + \omega B\cos\omega t$$

and

$$Y'' = -\omega^2 A\cos\omega t - \omega^2 B\cos\omega t$$

Plugging this into the left hand side of the original equation we get,

$$Y'' + \lambda Y' + Y = (A + \lambda\omega B - \omega^2 A)\cos\omega t + (B - \lambda\omega A - \omega^2 B)\sin\omega t$$

We would like this to equal the right hand side, i.e. $F_0 \sin\omega t$. Comparing like terms,

$$A + \lambda\omega B - \omega^2 A = 0$$

$$B - \lambda\omega A - \omega^2 B = F_0$$

Using the first equation, we write A in terms of B:

$$A = \frac{-\lambda\omega}{1-\omega^2}B$$

Plugging this into the second equation:

$$B + \frac{\lambda^2\omega^2}{1-\omega^2} - \omega^2 B = f_0$$

$$B[(1-\omega^2)^2 + \lambda^2\omega^2] = F_0(1-\omega^2)$$

Thus,

$$A = \frac{F_0(1-\omega^2)}{\lambda^2\omega^2 + (1-\omega^2)^2} \quad B = \frac{-F_0\lambda\omega}{\lambda^2\omega^2 + (1-\omega^2)^2}$$

(b) Find the amplitude of the steady state solution from part (a).

Solution: The amplitude is given by $R = \sqrt{A^2 + B^2}$. Computing this quantity gives

$$R = \frac{F_0}{\sqrt{\lambda^2\omega^2 + (1-\omega^2)^2}}$$

(c) Let $F_0 = 1$ and $\lambda = 1$. Plot the amplitude as a function of forcing frequency ω.

Solution:

(d) Discuss what happens to the above graph when λ gets closer and closer to 0.

Solution:

2. A spring is stretched 6 inches by a mass that weighs 8 lb. The mass is attached to a mechanism that has a damping constant of 0.25 lb s/ft and is acted upon by an external force of $f \cos 2t$ lb. Assume that acceleration due to gravity, $g = 32$ ft/s^2.

   Find the steady state solution for the system. Give the form of the transient solution (with arbitrary constants).

   Solution: The spring constant is $k = 8/(1/2) = 16$lb/ft and the mass is $8/32 = 0.25$ lb s^2/ft. The damping constant is $\gamma = 0.25$lbs/ft. The external force is $4\cos 2t$ lb. Therefore the equation of motion is

$$u'' + u' + 64u = 16\cos 2t.$$

   Solving this gives the homogeneous solution

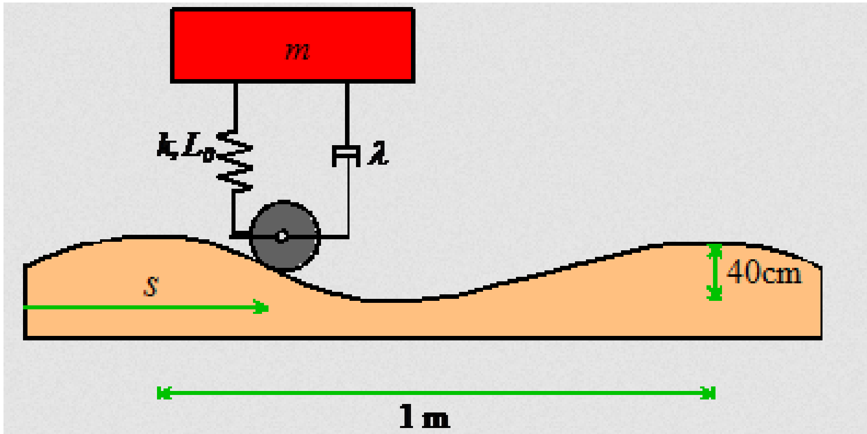
$$u_c(t) = e^{-t/2}[c_1\cos(\sqrt{255}t/2) + c_2\sin(\sqrt{255}t/2)]$$

   and particular solution

$$U(t) = (240/901)\cos(2t) + (8/901)\sin(2t).$$

   Since the homogeneous solution dies as $t \to 0$, the steady state solution is just the particular solution.

3. A car and its suspension system are idealized as a damped spring-mass system, with natural frequency 0.5Hz and damping coefficient 0.2. Suppose the car drives at speed V over a road with sinusoidal bumps. Suppose the distance between two bumps is 10m and the height of a bump 20cm. At what speed does the maximum amplitude vibration occur and what is the corresponding vibration amplitude? (see figure)



Solution: Let s denote the distance travelled by the car and let L denote the wavelength of the bumps and H the bump amplitude". Then the height of the wheel above the mean road height may be expressed as

$$y = H \sin\left(\frac{2\pi s}{l}\right).$$

Since the car has speed V, we have $s = Vt$, so,

$$y(t) = H \sin\left(\frac{2\pi V}{L} t\right),$$

i.e., the wheel oscillates vertically with sinusoidally at frequency $\frac{2\pi V}{L}$.

Now the suspension has been idealized as a spring mass system subject to base excitation. The diffeq is thus,

$$my'' + \gamma y' + ky' = H \sin\left(\frac{2\pi V}{L} t\right).$$

From the formula for amplitude, we have

$$R = \frac{H}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma\omega^2}}$$

where $\omega_0 = 0.5$Hz (natural frequency) and $\omega = \frac{2\pi V}{L}$ the forcing frequency. The maximum amplitude therefore, is achieved when $\omega = \omega_0$, i.e.,

$$\omega = \frac{2\pi V}{L} = \omega_0 \implies V = 5m/s.$$

The amplitude can be found through straightforward substitution to be 50cm.

Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee` and `www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_forced/vibrations_forced.htm`