§3.7,3.8 **Name:** ____________________

Fall MATH 2930 21 October 2021

Review

Free and Forced Vibrations

- The following equation represents vibrations:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t),$$

where m is *mass*, γ is the *damping coefficient*, k is the *spring constant* and $F(t)$ is the *forcing*.

- $\omega_0 = \sqrt{k/m}$ is referred to as the *natural frequency*.
- When $F(t) = 0$, i.e. homogeneous, we refer to the vibrations as *free*.
- Additionally, if $\gamma = 0$ we refer to the vibrations as *undamped*.
- The solution to undamped free vibrations is

$$u(t) = R\cos(\omega_0 t - \delta)$$

thus it oscillates with a constant amplitude forever.

- The solution to damped free vibrations when $0 < \gamma^2 < 4km$ (small damping) is

$$u(t) = R_0 e^{-\gamma t/(2m)} \cos(\omega_0 t - \delta_0)$$

thus the amplitude of oscillation decreases with time due to the exponential factor.

- The forcing, $F(t)$ is usually a sine or cosine when it is present, i.e., $F(t) = F_0\cos(\omega t)$ where ω is called the *forcing frequency*.
- If $\omega \neq \omega_0$, we have the general solution

$$u(t) = \underbrace{R_0 e^{-\gamma t/(2m)} \cos(\omega_0 t - \delta_0)}_{\text{homogeneous sol.}} + \underbrace{R\cos(\omega t - \delta)}_{\text{particular sol.}}$$

- If $\gamma = 0$ and $\omega = \omega_0$, we have *resonance* and

$$u(t) = R_0\cos(\omega_0 t - \delta_0) + R\,t\cos(\omega t - \delta)$$

(notice the factor of t in the second term). In this case the amplitude goes to ∞ as t increases.

Practice Problems

1. Example problem: Consider the damped, forced oscillator described by

$$u'' + \lambda u' + u = F_0 \sin(\omega t)$$

where $\lambda > 0$.

(a) Find the steady state (i.e. particular) solution of this equation.

(b) Find the amplitude of the steady state solution from part (a).

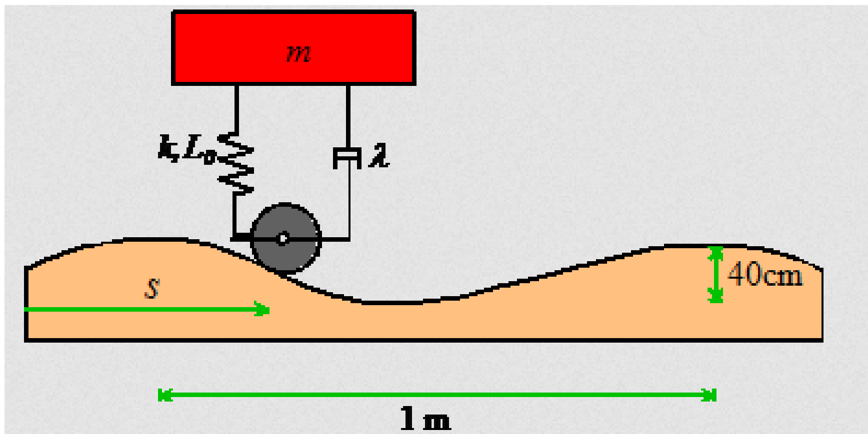
(c) Let $F_0 = 1$ and $\lambda = 1$. Plot the amplitude as a function of forcing frequency ω.

(d) Discuss what happens to the above graph when λ gets closer and closer to 0.

2. A spring is stretched 6 inches by a mass that weighs 8 lb. The mass is attached to a mechanism that has a damping constant of 0.25 lb s/ft and is acted upon by an external force of $f \cos 2t$ lb. Assume that acceleration due to gravity, $g = 32$ ft/s^2.

   Find the steady state solution for the system. Give the form of the transient solution (with arbitrary constants).

3. A car and its suspension system are idealized as a damped spring-mass system, with natural frequency 0.5Hz and damping coefficient 0.2. Suppose the car drives at speed V over a road with sinusoidal bumps. Suppose the distance between two bumps is 10m and the height of a bump 20cm. At what speed does the maximum amplitude vibration occur and what is the corresponding vibration amplitude? (see figure)



Problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee` and `www.brown.edu/Departments/Engineering/Courses/En4/Notes/vibrations_forced/vibrations_forced.htm`