§4.1,4.2,4.3 NAME: ____________________

Fall MATH 2930 28 October 2021

Review

Higher Order ODEs

- Very generally, an n^{th} order linear, constant coefficient ODE is written as

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = g(t).$$

  Here, $a_0, ..., a_n$ are numbers and $a_n \neq 0$.

- If $g(t) = 0$, we call the ODE *homogeneous*. Otherwise it is inhomogeneous/non-homogeneous.

- Just as before, to solve a homogeneous ODE, we first substitute e^{rt} to get the characteristic polynomial equation,

$$a_0 r^n + a_1 r^{n-1} + ... + a_{n-1} r + a_n = 0.$$

- The above equation has n roots (some may repeat). In general, it is difficult to solve this equation (factor the polynomial) exactly. If it is possible, and the roots are $r_1, ..., r_n$, we have:

  1. For each **non-repeated** root, r, then the solution corresponding to that root is ce^{rt}, where c is an arbitrary constant.
  2. Suppose the root r **repeats** s number of times, then the corresponding solution is $c_1 e^{rt} + c_2 t e^{rt} + ... + c_s t^s e^{rt}$.
  3. For **complex** roots (they always appear in conjugate pairs), $r_\pm = a \pm ib$ the corresponding root is $e^{at}(c_1 \cos bt + c_2 \sin bt)$.

- For inhomogeneous equations, the method of undetermined coefficients can be used to find a particular solution (just like in the 2^{nd} order case).

Practice Problems

1. Example problem: Find the general solution to the following differential equation:

$$y^{(4)} - 2y^{(3)} + 2y'' - 2y' + y = e^t + \cos 2t$$

Solution: We compute the characteristic equation as $r^4 - 2r^3 + 2r^2 - 2r + 1 = 0$. Observe that we can write the polynomial as

$$\begin{aligned} r^4 - 2r^3 + r^2 + r^2 - 2r + 1 &= r^2(r^2 - 2r + 1) + (r^2 - 2r + 1) \\ &= (r^2 - 2r + 1)(r^2 + 1) \\ &= (r-1)(r-1)(r-i)(r+i) \end{aligned}$$

Thus the complementary solution is

$$y_c = c_1 e^t + c_2 t e^t + c_3 \cos t + c_4 \sin t.$$

The first term of the right hand side is e^t. Since the root $r = 1$ is repeated twice, we multiply the assume $Y_1 = At^2 e^t$. Compute the derivatives:

$$\begin{aligned} Y_1' &= At^2 e^t + 2Ate^t \\ Y_1'' &= At^2 e^t + 4Ate^t + 2Ae^t \\ Y_1^{(3)} &= At^2 e^t + 6Ate^t + 6Ae^t \\ Y_1^{(4)} &= At^2 e^t + 8Ate^t + 12Ae^t \end{aligned}$$

Plugging this in to the inhomogeneous equation we get

$$4Ae^t = e^t \implies A = \frac{1}{4}.$$

For the second term we take $Y_2 = B\cos 2t + C\sin 2t$. The derivatives are

$$\begin{aligned} Y_2' &= -2B\sin 2t + 2C\cos 2t \\ Y_2'' &= -4B\cos 2t - 4C\sin 2t \\ Y_2^{(3)} &= 8B\sin 2t - 8C\cos 2t \\ Y_2^{(4)} &= 16B\cos 2t + 16C\sin 2t \end{aligned}$$

Plugging in the derivatives and simplifying we get $B = 9/225$ and $C = 12/225$. The solution is therefore

$$y = c_1 e^t + c_2 t e^t + \frac{1}{4} t^2 e^t + c_3 \cos t + c_4 \sin t + \frac{1}{225}(9\cos 2t + 12\sin 2t)$$

2. We are given that the homogeneous solution to a linear constant coefficient ODE is

$$y = c_1 e^t + c_2 t e^t + c_3 t^2 e^t + c_4 e^{-t} + e^t(c_5 \cos t + c_6 \sin t).$$

Find the differential equation.

Solution: By observing the solution, we see that the roots of the equation are $1, 1, 1, -1, 1+i, 1-i,$. The characteristic equation has to be

$$\begin{aligned}(r-1)^3(r+1)(r-1-i)(r-1+i) &= (r^3 - 3r^2 + 3r - 1)(r+1)(r^2 - 2r + 2) \\ &= r^6 - 4r^5 + 6r^4 - 2r^3 - 5r^2 + 6r - 2\end{aligned}$$

The differential equation is therefore

$$y^{(6)} - 4y^{(5)} + 6y^{(4)} - 2y^{(3)} - 5y'' + 6y' - y = 0.$$

3. Find the general solution to the following ODEs:

(a) $y^{(4)} + 2y'' + y = 0$

Solution: The characteristic equation is $r^4 + 2r^2 + 1 = (r^2+1)^2 = 0$. The roots are $r = \pm i, \pm i$ (repeated). So the solution is

$$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$$

(b) $y^{(6)} - y'' = 0$.

Solution: The characteristic equation is $r^6 - r^2 = 0$. We can factor it as follows:

$$\begin{aligned} r^6 - r^2 &= r^2(r^4 - 1) \\ &= r^2(r^2+1)(r^2-1) \\ &= r^2(r^2+1)(r+1)(r-1) = 0 \end{aligned}$$

So the roots are $r = 0, 0, \pm i, 1, -1$. The corresponding general solution is

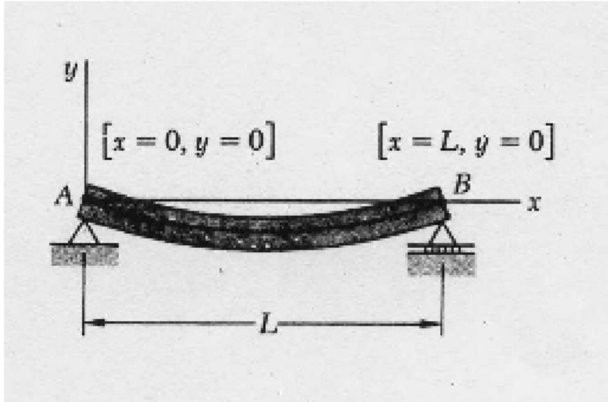
$$y = c_1 + c_2 t + c_3 \cos t + c_4 \sin t + c_5 e^t + c_6 e^{-t}.$$

4. Consider a horizontal metal beam of length L subject to a vertical load $f(x)$ per unit length. The resulting vertical displacement in the beam $y(x)$ satisfies a differential equation of the form

$$A \frac{d^4 y}{dx^4} = f(x)$$

where A is a constant. Suppose that $f(x)$ is just a constant k, i.e., the equation is simply

$$A \frac{d^4 y}{dx^4} = k.$$



(a) Find the general solution of this non-homogeneous fourth order equation.

Solution: Although we could solve this equation in the usual way using undetermined coefficients, we can simply integrate the equation 4 times:

$$y = \frac{kx^4}{24A} + C_1x^3 + C_2x^2 + C_3x + C_4$$

(b) Solve the above equation for the boundary condition

$$y(0) = y''(0) = y(L) = y''(L) = 0.$$

Solution: The derivatives of the solution are

$$\begin{aligned} y' &= \frac{kx^3}{6A} + 3C_1x^2 + 2C_2x + C_3 \\ y'' &= \frac{kx^2}{2A} + 6C_1x + 2C_2 \\ y''' &= \frac{kx}{A} + 6C_1 \end{aligned}$$

Plugging in y and y'' into the boundary condition:

$$\begin{aligned} y(0) &= C_4 = 0 \\ y''(0) &= 2C_2 = 0 \\ y(L) &= \frac{kL^4}{24A} + C_1L^3 + C_2L^2 + C_3L + C_4 = 0 \\ y''(L) &= \frac{kL^2}{2A} + 6C_1L + 2C_2 = 0 \end{aligned}$$

The first two equations just say that $C_2 = C_4 = 0$. This simplifies the other two equations:

$$\frac{kL^4}{24A} + C_1L^3 + C_3L = 0$$
$$\frac{kL^2}{2A} + 6C_1L = 0$$

Solving the second equation gives

$$C_1 = \frac{-kL}{12A}$$

and solving C_3 gives

$$C_3 = \frac{kL^3}{24A}$$

Plugging in all the constants,

$$y = \frac{k}{24A}(x^4 - 2Lx^3 + L^3x)$$

(c) Solve the above equation for the boundary condition

$$y(0) = y'(0) = y''(L) = y'''(L) = 0.$$

Solution: Plugging in the derivatives into the boundary conditions:

$$\begin{aligned} y(0) &= C_4 = 0 \\ y'(0) &= C_3 = 0 \\ y''(L) &= \frac{kL^2}{2A} + 6C_1L + 2c_2 = 0 \\ y'''(L) &= \frac{kl}{A} + 6C_1 = 0 \end{aligned}$$

Solving, we get $C_1 = \frac{-kL}{6A}$, $C_2 = \frac{kL^2}{4A}$ so the solution is

$$y = \frac{k}{24A}(x^4 - 4Lx^3 + 6L^2x^2).$$

Some problems in this worksheet are adapted from `https://math.uchicago.edu/~ecartee`