PRELIM REVIEW   NAME: ______________________

Fall MATH 2930   04 November 2021

1. Consider the following differential equation:

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = g(t)$$

   (a) Find the homogeneous/complementary solution (i.e. $g(t) = 0$)

   Solution: The homogeneous solution is $y_c(t) = Ae^t \cos 2t + Be^t \sin 2t$.

   (b) Suppose $g(t) = e^t \sin(2t)$. Find the particular solution. What is the steady state solution?

   Solution: We guess the particular solution to be

$$Y = te^t[A \sin 2t + B \cos 2t]$$

   The derivatives are

$$Y' = te^t[(A - 2b) \sin 2t + (2A + B) \cos 2t] + e^t[A \sin 2t + B \cos 2t]$$

$$Y'' = te^t[(-3A - 4B) \sin 2t + (4A - 3B) \cos 2t] + e^t[-4B \sin 2t + (4A + 2B) \cos 2t]$$

   Plugging in the particular solution into the inhomogeneous solution, we get

$$e^t[(-4B - 2A) \sin 2t + (4A) \cos 2t] = e^t \sin 2t$$

   Thus $-4B - 2A = 1$ and $4A = 0$ i.e. $A = 0$ and $B = -1/4$. The particular solution is

$$Y = -\frac{1}{4}te^t \cos 2t$$

   (c) What would the guess for the particular solution be if $g(t) = e^{3t} \sin(2t)$ instead? (**No need to find the undetermined coefficients**).

   Solution: In this case, we simply make the guess $Y = e^{3t}(A \sin 2t + B \cos 2t)$.

2. Consider the two coupled differential equations:

$$u_1'' + 5u_1 = 2u_2 \qquad u_2'' + 2u_2 = 2u_1$$

   Here, ‘coupled’ means that the differential equation for u_1 has u_2 terms and vice versa.

   (a) Express the two 2nd order equations above as a single 4th order equation involving only u_1. *Hint: Try to express u_2 in terms of u_1 using the second equation and plug it into the first.*

Solution: $u_1^{(4)} + 27u_1'' + 6u_1 = 0$.

(b) Find the general solution to the 4th order equation you found above with the initial conditions

$$u_1(0) = 1 \quad u_1'(0) = 0 \quad u_2(0) = 2 \quad u_2'(0) = 0.$$

Solution: Roots: $\pm i$ and $\pm\sqrt{6}i$. So $u_1(t) = A\cos t + B\sin t + C\cos\sqrt{6}t + D\sin\sqrt{6}t$.

$$u_1'(t) = -A\sin t + B\cos t - \sqrt{6}C\sin\sqrt{6}t + \sqrt{6}D\cos\sqrt{6}t$$

$$u_1''(t) = -A\cos t - B\sin t - 6C\cos\sqrt{6}t - 6D\sin\sqrt{6}t$$

Compute u_2 and u_2':

$$u_2(t) = \frac{1}{2}(4A\cos t + 4B\sin t - C\cos\sqrt{6}t - D\sin\sqrt{6}t)$$

$$u_2'(t) = \frac{1}{2}(-4A\sin t + 4B\cos t + \sqrt{6}C\sin\sqrt{6}t - \sqrt{6}D\cos\sqrt{6}t)$$

Plugging in the initial conditions:

$$A + C = 1$$

$$B + \sqrt{6}D = 0$$

$$2A - \frac{C}{2} = 2$$

$$4B - \sqrt{6}D = 0$$

Solving these, we get $A = 1$, $B = C = D = 0$.

3. Assume that the system described by the equation

$$mu'' + \gamma u' + ku = 0$$

is critically damped and that the initial conditions are $u(0) = u_0$, $u'(0) = v_0$.

(a) If $v_0 = 0$ and $u_0 \neq 0$ show that $u \to 0$ as $t \to \infty$ but that u is never zero.
Solution: $u(t) = c_1 e^{-\gamma/2mt} + c_2 t e^{-\gamma/2mt}$. Using the initial conditions

$$u(t) = u_0 e^{-\gamma/2mt} + \left(v_0 + \frac{\gamma}{2m}u_0\right) t e^{-\gamma/2mt}$$

If $v_0 = 0$, the solution simplifies to

$$u(t) = u_0\left(1 + \frac{\gamma}{2m}t\right) e^{-\gamma/2mt}$$

Then

$$\lim_{t\to\infty} u(t) = 0 \qquad \text{(Lopital's rule)}$$

However, for any finite t, $u(t) \neq 0$. This is because the exponential term is always positive and $\left(1 + \frac{\gamma}{2m}t\right) \neq 0$ for any non-zero t.

(b) Assuming $u_0 > 0$, determine a condition on v_0 that will ensure that the mass passes through its equilibrium once released.

Solution: Recall that

$$u(t) = u_0 e^{-\gamma/2mt} + \left(v_0 + \frac{\gamma}{2m}u_0\right) te^{-\gamma/2mt}$$

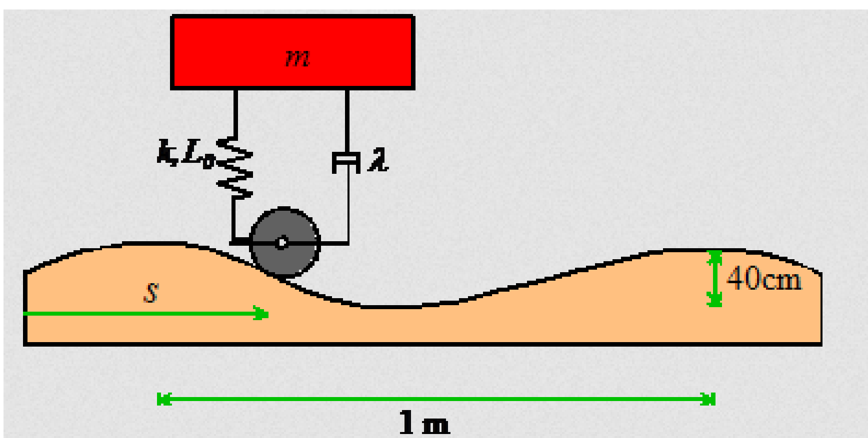
Factor out the exponential so that

$$u(t) = \left[u_0 + \left(v_0 + \frac{\gamma}{2m}u_0\right) t\right] e^{-\gamma/2mt}$$

We want a condition so that there is some $\tau > 0$ for which $u(\tau) = 0$. Since the exponential is always positive, the only way this is possible is if

$$u_0 + \left(v_0 + \frac{\gamma}{2m}u_0\right) \tau = 0 \implies \tau = \frac{-u_0}{v_0 + \frac{\gamma}{2m}u_0}$$

4. A car and its suspension system are idealized as a damped spring-mass system, with natural frequency 0.5Hz and damping coefficient 0.2. Suppose the car drives at speed V over a road with sinusoidal bumps. Suppose the distance between two bumps is 10m and the height of a bump 20cm. At what speed does the maximum amplitude vibration occur and what is the corresponding vibration amplitude? (see figure)



Solution: Let s denote the distance travelled by the car and let L denote the wave-length of the bumps and H the bump amplitude". Then the height of the wheel above the mean road height may be expressed as

$$y = H \sin\left(\frac{2\pi s}{l}\right).$$

Since the car has speed V, we have $s = Vt$, so,

$$y(t) = H \sin\left(\frac{2\pi V}{L}t\right),$$

i.e., the wheel oscillates vertically with sinusoidally at frequency $\frac{2\pi V}{L}$.

Now the suspension has been idealized as a spring mass system subject to base excitation. The diffeq is thus,

$$my'' + \gamma y' + ky' = H \sin\left(\frac{2\pi V}{L} t\right).$$

From the formula for amplitude, we have

$$R = \frac{H}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma\omega^2}}$$

where $\omega_0 = 0.5$Hz (natural frequency) and $\omega = \frac{2\pi V}{L}$ the forcing frequency. The maximum amplitude therefore, is achieved when $\omega = \omega_0$, i.e.,

$$\omega = \frac{2\pi V}{L} = \omega_0 \implies V = 5m/s.$$

The amplitude can be found through straightforward substitution to be 50cm.