§10.1,10.2,10.3 NAME: ____________________

Fall MATH 2930 11 November 2021

Review

Two Point Boundary Value Problems

- A two point boundary value problem (bvp) is typically of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(\alpha) = y_0 \quad y(\beta) = y_1$$

  where α and β are two points on the domain.

- The important thing to note is that instead of specifying y and y' at the **same** point, we specify **only** y but at **two different** points.

- A 2-point bvp is *homogeneous*, if $g(t) = 0$ and $y_0 = y_1 = 0$, otherwise it is *non-homogeneous* (note that unlike before, the definition also depends on the boundary values).

- Consider the homogeneous 2 point bvp

$$y'' + \lambda y = 0, \quad y(0) = 0 \quad y(\pi) = 0$$

- $y = 0$ is called a *trivial solution* and every other solution is called *nontrivial*.

- If for some value of λ, there is a *nontrivial* solution y, then we say that λ is an *eigenvalue* and y is an *eigenfunction*.

- If $\lambda > 0$, the eigenvalues are n^2 and the eigenfunctions are $\sin(nx)$ respectively (n is a natural number). Thus there are infinitely many eigenvalues and eigenfunctions

- If $\lambda \leq 0$, there are no eigenvalues or eigenvectors.

Fourier Series

- A function, f is *periodic* with *period* $T > 0$ if for any x:

$$f(x + T) = f(x).$$

- The *fundamental period* of a periodic function is the smallest T with the above property.

- Let f and g be two functions on $[-L, L]$. The *inner product* of f and g, denoted (f, g) is

$$(f, g) = \int_{-L}^{L} f(x)g(x)dx$$

- If $(f, g) = 0$, we say that f and g are *orthogonal*.
- The functions $\cos\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{m\pi}{L}x\right)$ are very important to us.
- $\cos\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$ are orthogonal (except when $m = n$):

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

- $\sin\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$ are orthogonal (except when $m = n$):

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

- $\cos\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$ are orthogonal for all m and n:

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0$$

- Let f be a function on $[-L, L]$. Its *Fourier series* is given by

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left[a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right]$$

- the constants, a_n, b_k with $n = 0, 1, 2, ...$ and $k = 1, 2, 3, ...$ are given by:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \quad b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Practice Problems

1. Find the first four terms in the Fourier series (a_0, b_1, a_1, b_2) of the following functions on the interval $[-\pi, \pi]$:

   (a) $f(x) = 1$ (constant function)

   Solution: Since the function is defined on $[-\pi, \pi]$, we have $L = \pi$. Let's compute the coefficients:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1dx = \frac{1}{\pi}2\pi = 2.$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\sin\left(\pi x/\pi\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin x dx = -\frac{1}{\pi}[\cos(\pi) - \cos(-\pi)] = 0$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\cos\left(\pi x/\pi\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos x dx = \frac{1}{\pi}[\sin(\pi) - \sin(-\pi)] = 0$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\sin\left(2\pi x/\pi\right)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin 2x dx = -\frac{1}{2\pi}[\cos(2\pi) - \cos(-2\pi)] = 0$$

   So the Fourier series is simply 1 (as expected).

   (b) $f(x) = x$

   Solution: Compute the coefficients

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{pi} x dx = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin\left((\pi x/\pi\right)dx = 2$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\cos\left(\pi x/\pi\right)dx = 0$$

$$b_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} 1\sin\left(\pi x/\pi\right)dx = -1$$

   So the Fourier series is $2\sin(x) - \sin(2x) + \ldots$.

   (c) $f(x) = \cos(x)$

   Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{pi}\cos(x)dx = 0$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{pi}\cos(x)\sin(x)dx = \frac{1}{2\pi}\int_{-\pi}^{pi}\sin(2x)dx = 0$$

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_{-\pi}^{pi} \cos(x)\cos(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{pi} \frac{1+\cos(2x)}{2}dx \\
&= \frac{1}{2\pi}\left[\pi + \frac{1}{2}\int_{-\pi}^{\pi}\cos(2x)dx\right] \\
&= 1
\end{aligned}
$$

$$
b_2 = \frac{1}{\pi}\int_{-\pi}^{pi}\cos(x)\sin(2x)dx = \frac{1}{2\pi}\int_{\pi}^{\pi}\sin(3x)+\sin(x)dx = 0
$$

So, as expected the Fourier series is simply $\cos(x)$.

2. Either solve the bvp

$$
y'' + 4y = \cos x \quad y'(0) = 0 \quad y'(\pi) = 0
$$

or show that it has no solution. (Notice that instead of specifying y at the two points, we specified y', but this is still a 2-point bvp)

Solution: By using undetermined coefficients, we get the following general solution

$$
y(x) = c_1\cos(2x) + c_2\sin(2x) + \frac{1}{3}\cos(x).
$$

Differentiating yields,

$$
y'(x) = -2c_1\sin(2x) + 2c_2\cos(2x) - \frac{1}{3}\sin(x).
$$

Now,

$$
y'(0) = -2c_1\sin(0) + 2c_2\cos(0) - \frac{1}{3}\sin(0) = 2c_2 = 0,
$$

so that $c_2 = 0$. The second boundary condition gives

$$
y'(\pi) = -2c_1\sin(2\pi) - \frac{1}{3}\sin(2\pi) = 0
$$

The above is true regardless o the value of c_1, hence we have infinitely many solutions given by

$$
y(x) = c_1\cos(2x) + \frac{1}{3}\cos(x).
$$

3. Find all the eigenvalues and eigenfunctions of the bvp:

$$
y'' + \lambda y = 0 \quad y'(0) = 0 \quad y'(\pi) = 0.
$$

(assuming all eigenvalues are real).

Solution: We focus on three regimes, $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

$\lambda = 0$:

We have the equation $y'' = 0$, and integrating twice gives $y(x) = c_1 + c_2 x$, where c_1 and c_2 are constants of integration. The derivative is simply $y'(x) = c_2$. By enforcing $y'(0) = 0$, we get $c_2 = 0$. The second boundary condition tells us nothing interesting, so the eigenfunctions are $y(x) = c_1$ ($c_1 \neq 0$). We avoid $c_1 = 0$, since this would be a trivial solution. The corresponding eigenvalue is 0.

$\lambda > 0$:

For convenience we say $\lambda = \mu^2$, $\mu \neq 0$. Then our differential equation is

$$y'' + \mu^2 y = 0.$$

The general solution is the all too familiar

$$y(x) = A\cos(\mu x) + B\sin(\mu x).$$

Taking the derivative yields

$$y'(x) = -A\mu\sin(\mu x) + B\mu\cos(\mu x).$$

Enforcing the first boundary condition yields,

$$y'(0) = 0 = -A\mu\sin(0) + B\mu\cos(0) = B\mu.$$

therefore $B = 0$ (since we have assumed that $\mu \neq 0$). The boundary condition gives

$$y'(\pi) = 0 = -A\mu\sin(\mu\pi).$$

We want non-trivial solutions, so instead of taking $c_5 = 0$, we can look at values of μ for which $\sin(\mu x) = 0$. This happens for any integer μ, so the eigenvalues are $\lambda = \mu^2 = 1, 4, 9, 16, ...$ and the corresponding eigenfunctions are $\cos(nx)$.

$\lambda < 0$:

Let's write $\lambda = -\mu^2$, $\mu \neq 0$. Then we have

$$y'' - \mu^2 y = 0.$$

The general solution is of the form

$$y(x) = Ce^{-\mu x} + De^{\mu x}$$

and derivative

$$y'(x) = -C\mu e^{\mu x} + D\mu e^{\mu x}.$$

The first boundary condition says that $y'(0) = 0 = -C\mu + D\mu$ thus $C = D$. The second boundary condition says $y'(\pi) = D\mu(e^{\mu\pi} - e^{-\mu\pi})$. Since the term in brackets cannot be zero, C must be equal to zero. Thus, $\lambda < 0$ cannot be an eigenvalue since the all the solutions are trivial.