§10.1,10.2,10.3 Name: ____________________

Fall MATH 2930 11 November 2021

Review

Two Point Boundary Value Problems

- A two point boundary value problem (bvp) is typically of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

$$y(\alpha) = y_0 \quad y(\beta) = y_1$$

where α and β are two points on the domain.

- The important thing to note is that instead of specifying y and y' at the **same** point, we specify **only** y but at **two different** points.

- A 2-point bvp is *homogeneous*, if $g(t) = 0$ and $y_0 = y_1 = 0$, otherwise it is *non-homogeneous* (note that unlike before, the definition also depends on the boundary values).

- Consider the homogeneous 2 point bvp

$$y'' + \lambda y = 0, \quad y(0) = 0 \quad y(\pi) = 0$$

- $y = 0$ is called a *trivial solution* and every other solution is called *nontrivial*.

- If for some value of λ, there is a *nontrivial* solution y, then we say that λ is an *eigenvalue* and y is an *eigenfunction*.

- If $\lambda > 0$, the eigenvalues are n^2 and the eigenfunctions are $\sin(nx)$ respectively (n is a natural number). Thus there are infinitely many eigenvalues and eigenfunctions

- If $\lambda \leq 0$, there are no eigenvalues or eigenvectors.

Fourier Series

- A function, f is *periodic* with *period* $T > 0$ if for any x:

$$f(x + T) = f(x).$$

- The *fundamental period* of a periodic function is the smallest T with the above property.

- Let f and g be two functions on $[-L, L]$. The *inner product* of f and g, denoted (f, g) is

$$(f, g) = \int_{-L}^{L} f(x)g(x)dx$$

- If $(f,g)=0$, we say that f and g are *orthogonal*.
- The functions $\cos\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{m\pi}{L}x\right)$ are very important to us.
- $\cos\left(\frac{m\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$ are orthogonal (except when $m=n$):

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx=\begin{cases}0 & m\neq n\\ L & m=n\end{cases}$$

- $\sin\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$ are orthogonal (except when $m=n$):

$$\int_{-L}^{L}\sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx=\begin{cases}0 & m\neq n\\ L & m=n\end{cases}$$

- $\cos\left(\frac{m\pi}{L}x\right)$ and $\sin\left(\frac{n\pi}{L}x\right)$ are orthogonal for all m and n:

$$\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx=0$$

- Let f be a function on $[-L,L]$. Its *Fourier series* is given by

$$f(x)=\frac{a_0}{2}+\sum_{m=1}^{\infty}\left[a_m\cos\left(\frac{m\pi}{L}x\right)+b_m\sin\left(\frac{m\pi}{L}x\right)\right]$$

- the constants, a_n, b_k with $n=0,1,2,...$ and $k=1,2,3,...$ are given by:

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi}{L}x\right)dx\quad b_k=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

Practice Problems

1. Find the first four terms in the Fourier series (a_0, b_1, a_1, b_2) of the following functions on the interval $[-\pi, \pi]$:

   (a) $f(x) = 1$ (constant function)

   (b) $f(x) = x$

(c) $f(x) = \cos(x)$

2. Either solve the bvp

$$y'' + 4y = \cos x \quad y'(0) = 0 \quad y'(\pi) = 0$$

or show that it has no solution. (Notice that instead of specifying y at the two points, we specified y', but this is still a 2-point bvp)

3. Find all the eigenvalues and eigenfunctions of the bvp:

$$y'' + \lambda y = 0 \quad y'(0) = 0 \quad y'(\pi) = 0.$$

(assuming all eigenvalues are real).