§10.3,10.4,10.5 **Name:** ______________________

Fall MATH 2930 18 November 2021

Review

Fourier Series (contd.)

- f is a *piece-wise continuous* on $[a, b]$ if it can be broken into finitely many 'pieces' each of which is continuous. This kind of function can have 'jumps'.
- Fourier convergence theorem: If f is piece-wise continuous on $[-L, L]$ and it is periodic with period $2L$ outside $[-L, L]$, then the Fourier series (evaluated at x) converges to $f(x)$ wherever f is continuous. Wherever f has a jump, the Fourier series converges to the midpoint of the jump.
- Even function: $f(-x) = f(x)$ Odd function: $f(-x) = -f(x)$
- If f is even on $[-L, L]$, the Fourier series will only have cosine terms (and the constant term since it is technically also a cosine term):

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

- If f is odd on $[-L, L]$, the Fourier series will only have sine terms:

$$f(x) = \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{L}\right).$$

- Other facts about even and odd functions:
  - Sums and differences of even (odd) functions are even (odd),
  - Product (or quotient) of two even functions (two odd functions) is even,
  - Product (or quotient) of an even and odd function is odd.
- If f is even on $[-L, L]$, then $\int_{-L}^{L} f(x)dx = 2\int_0^L f(x)dx$.

  If f is odd on $[-L, L]$, then $\int_{-L}^{L} f(x)dx = 0$.
- Periodic extensions of functions: Let $f(x)$ be defined on $[0, L]$
  1. Even function with period $2L$:

$$f_{\text{even}} = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

2. Odd function with period $2L$:

$$f_{\text{odd}} = \begin{cases} f(x) & 0 < x < L \\ 0 & x = 0, L \\ -f(-x) & -L < x < 0 \end{cases}$$

The Heat Equation

- The heat equation has the form:

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < L, \quad t > 0$$

  where α^2 is a constant called *thermal diffusivity*.

- At $t = 0$, we prescribe an *initial temperature distribution*:

$$u(x, 0) = f(x), \quad 0 \le x \le L$$

  and we also have a 2-point boundary condition in x:

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0.$$

- Separation of Variables: we make the guess that the solution to the above equation is of the form

$$u(x, t) = X(x)T(t)$$

- Plugging in and solving the resulting 2-point boundary value problem gives

$$u_n(x, t) = e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3, ...$$

- The general solution is an infinite series:

$$u(x, t) = \sum_{n=1}^{\infty} c_n u_n(x, t)$$

  where

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Practice Problems

1. Indicate whether the following functions are even, odd or neither:

   (a) $\sin(x)$ on $[-\pi, \pi]$

   (b) $\cos(x)$ on $[-\pi, \pi]$

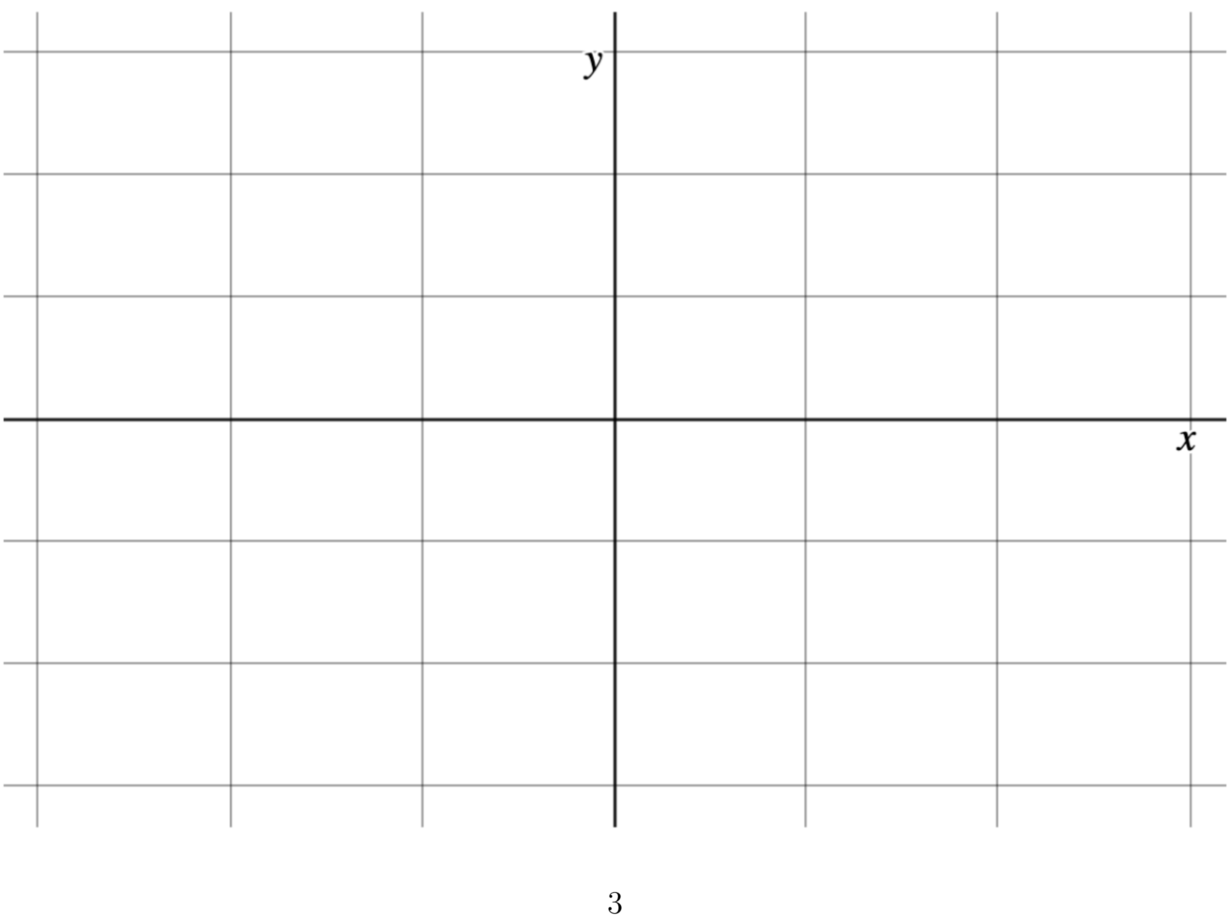
   (c) e^x on $[-5, 5]$

   (d)
   $$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ -x^2 & -1 \le x < 0 \end{cases}$$

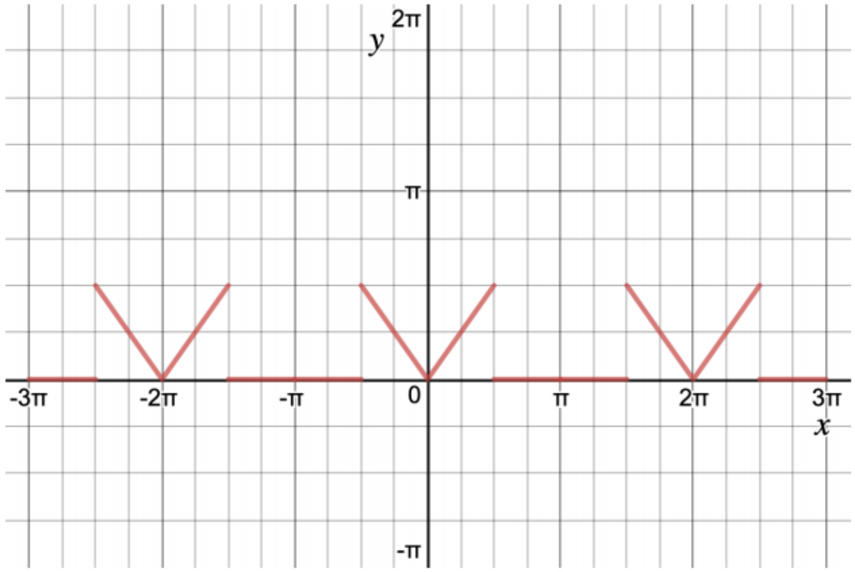
   (e)
   $$f(x) = \begin{cases} x^2 + 1 & 0 \le x \le 1 \\ -x^2 - 1 & -1 \le x < 0 \end{cases}$$

2. A function is defined on $[0, \pi]$ by
   $$f(x) = \begin{cases} x & 0 \le x \le \pi/2 \\ 0 & \pi/2 < x \le \pi \end{cases}$$

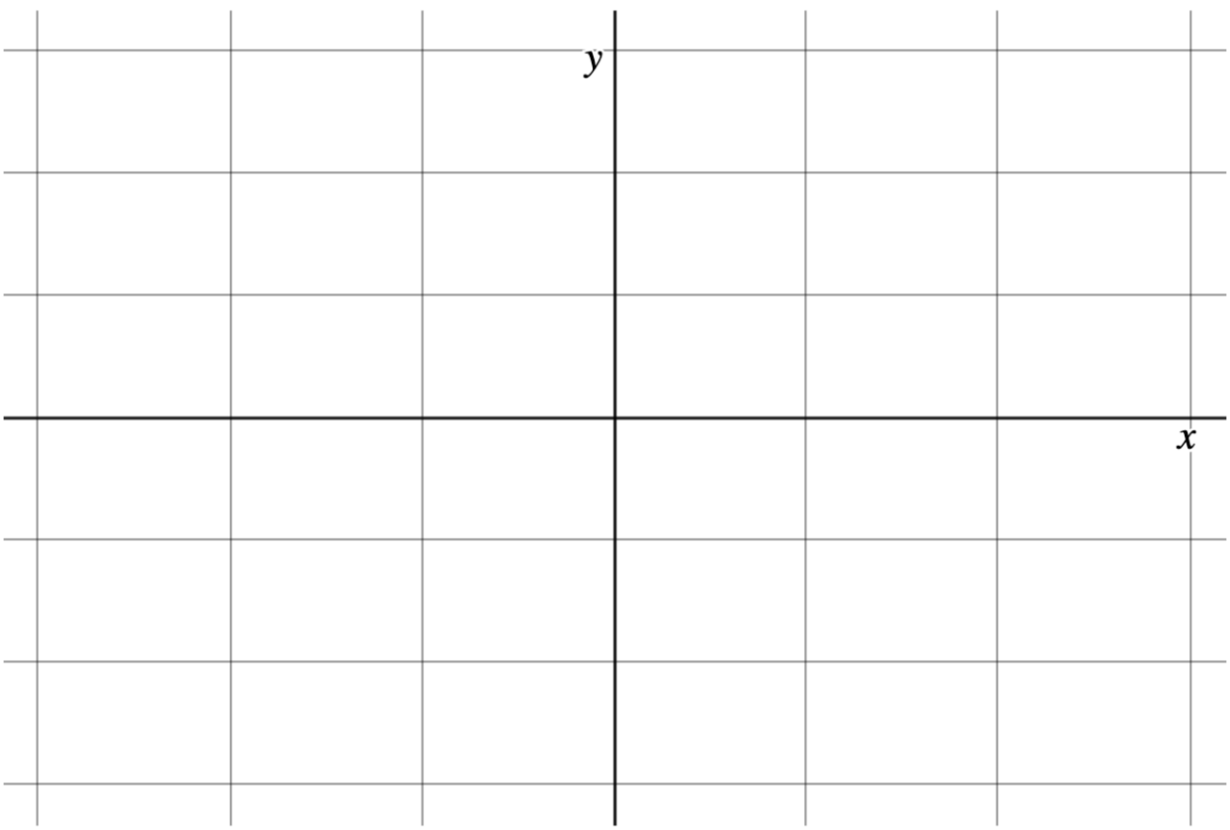
   (a) Sketch the **even** periodic extension of $f(x)$ on the interval $(-3\pi, 3\pi)$ on the axes below. Label important points on the x and y axes:



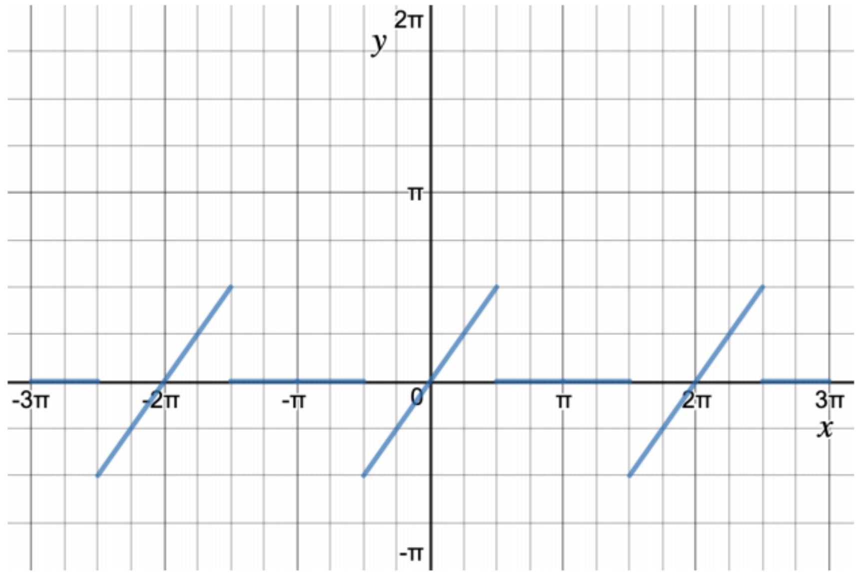
Solution:



(b) Sketch the **odd** periodic extension of $f(x)$ on the interval $(-3\pi, 3\pi)$ on the axes below. Label important points on the x and y axes:



Solution:



(c) Without doing any calculations, what value does the Fourier Cosine series of $f(x)$ converge to at $x = 3\pi/2$?

Solution: The Fourier Cosine series converges to the even periodic extension of the function. We notice that the function has a discontinuity at $3\pi/2$, thus the series converges to the average value of the jump i.e.,

$$\frac{0 + \pi/2}{2} = \frac{\pi}{4}$$

(d) $f(x)$ can be written as a Fourier Sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx).$$

Find the coefficients, b_n.

Solution: To compute the sine series, we use the odd extension:

$$f(x) = \begin{cases} 0 & -\pi \le x \le \frac{-\pi}{2} \\ x & \frac{-\pi}{2} \le x \le \frac{\pi}{2} \\ 0 & \frac{-\pi}{2} \le x < \pi \end{cases}$$

Let's compute the coefficients:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} x \sin(nx) dx = \int_{0}^{\pi/2} x \sin(nx) dx$$

where in the last step we used the fact that the product of two odd functions (x and $\sin(nx)$) is even.

Integrating by parts, we get

$$b_n = -\frac{1}{n}\cos(n\pi/2) + \frac{2}{n^2\pi}\sin(n\pi/2).$$

3. **Heat equation with insulated ends:**

Consider a thin pipe placed along the x-axis with ends at $x = 0$ and $x = \pi$. The pipe is filled with water and a small amount of a certain chemical. The chemical spreads (diffuses) through the pipe and the concentration of the chemical at location x and time t denoted $u(x,t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Initially the concentration has the following distribution

$$u(x,0) = x \quad 0 \leq x \leq \pi$$

The ends of the pipe are closed, so the chemical cannot escape. This can be written as

$$u_x(0,t) = 0 \quad u_x(\pi,t) \quad t \geq 0$$

(a) Assume that $u(x,t) = X(x)T(t)$ and find ODEs satisfied by X and T.

Solution:

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side depends only on t and the left depends only on x, both have to equal a constant, λ:

$$X''(x) - \lambda X(x) = 0$$

$$T'(t) - \lambda T(t) = 0$$

(b) Use the boundary conditions for u to derive boundary conditions for $X(x)$.

Solution:

$$u_x(0,t) = u_x(\pi,t) = 0$$

so we have

$$X'(0)T(t) = 0 \quad X'(\pi)T(t) = 0$$

If $T(t) = 0$ everywhere, the solution would be trivial, so we assume this is not the case. Thus it must be that

$$X'(0) = X'(\pi) = 0.$$

(c) Solve the resulting eigenvalue problem for $X(x)$.

Solution: Let's break down the problem into three different cases.

$\lambda > 0$:

In this case, the general solution is

$$X(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x),$$

Plugging in the initial conditions in this case simply gives $C_1 = C_2 = 0$.

$\lambda = 0$:

In this case, we get $X(x) = C_1 x + C_2$. Plugging in the initial conditions we get $C_1 = 0$ and arbitrary C_2 (but we can take $C_2 = 1$). So the eigenvalue is $\lambda_0 = 0$ and the eigenfunction is $X_0 = 1$.

$\lambda < 0$:

Let $\lambda = -\mu^2$ and we get

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x).$$

Plugging in the initial conditions we get $C_2 = 0$ and $\sin(\mu\pi) = 0$. Thus $\mu = n$ i.e. the eigenvalues are $\lambda_n = -n^2$ and the corresponding eigenfunctions are $X_n = \cos(nx)$.

(d) For each eigenvalue you found, solve the corresponding ODE for T.

Solution: For $\lambda_0 = 0$, the ODE for T_0 is given by

$$T_0'(t) = 0$$

We denote the solution by T_0 since it corresponds to λ_0. We thus have $T_0 = C_0$ for some constant C_0.

For $\lambda_n = -n^2$, the ODE is $T_n' = -n^2 T_n$ so $T_n = C_n e^{-n^2 t}$, for some constant C_n.

(e) Take linear combinations of all the fundamental solutions $u_n(x,t)$ to get the general solution $u(x,t)$ of this heat equation.

Solution: For λ_0, $u_0(x,t) = C_0/2$ (the factor of half is optional, but it simplifies a calculation in the next part). For λ_n, $u_n(x,t) = C_n \cos(nx) e^{-n^2 t}$ and

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) e^{-n^2 t}.$$

(f) Finally, use the initial condition to find the coefficients C_n.

Solution: Plug in $t = 0$ into the general solution:

$$u(x,0) = x = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) \quad x \in [0,\pi]$$

This is just the cosine series expansion for $f(x) = x$. We need to evenly extend $f(x)$ on $[-\pi, \pi]$, so we define $f(x) = -x$ for $x \in [-\pi, 0)$. In other words, we take $f(x) = |x|$ on $[-\pi, \pi]$. We can now compute the coefficients:

$$C_0 = \frac{2}{\pi} \int_0^\pi x dx = \pi$$

$$C_n = \frac{2}{\pi} \int_0^\pi x \cos(nx) dx = \frac{2}{n^2 \pi} [(-1)^n - 1].$$

Thus the final answer is

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [(-1)^n - 1] \cos(nx) e^{-n^2 t}$$