§10.6,10.7 **Name:** ____________________

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Review

Heat Equation (contd.)

- Non-homogeneous boundary conditions:

$$u(0,t) = T_1 \quad u(L,t) = T_2$$

General solution:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - (T_2 - T_1)\frac{x}{L} - T_1\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Insulated ends:

$$u_x(0,t) = 0 \quad u_x(L,t) = 0$$

General solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right) \qquad c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

Wave Equation

- The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

- Boundary conditions (fixed ends):

$$u(0,t) = 0 \quad u(L,t) = 0 \quad \text{for } t \geq 0$$

- Non-zero initial displacement but zero initial velocity:

$$u(x,0) = f(x) \quad u_t(x,0) = 0 \quad \text{for } 0 < x < L$$

General solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right) \qquad c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

- Zero initial displacement but non-zero initial velocity:

$$u(x,0) = 0 \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < L$$

$$u(x,t) = \sum_{n}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right) \qquad \frac{n\pi a}{L}k_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Laplace Equation

- The 2D Laplace's equation is given in rectangular (Cartesian) coordinates by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and in polar coordinates by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- Dirichlet problem on a rectangular region: $0 < x < a$ and $0 < y < b$ with the boundary conditions

$$\begin{aligned} &u(x,0) = 0, \quad u(x,b) = 0 && 0 < x < a \\ &u(0,y) = 0, \quad u(a,y) = f(y) && 0 < y < b \end{aligned}$$

General solution:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right) dy$$

- Dirichlet problem on a disk: $r < a$ and $0 \le \theta < 2\pi$ with the boundary condition

$$u(a,\theta) = f(\theta) \quad 0 \le \theta < 2\pi.$$

where f is periodic i.e. $f(0) = f(2\pi)$ (this vaguely acts like a boundary condition in the θ variable).

General solution:

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta))$$

$$c_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \quad n = 0, 1, 2, ...$$

$$k_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \quad n = 1, 2, ...$$

Practice Problems

1. **Heat equation with insulated ends:**
Consider a thin pipe placed along the x-axis with ends at $x = 0$ and $x = \pi$. The pipe is filled with water and a small amount of a certain chemical. The chemical spreads (diffuses) through the pipe and the concentration of the chemical at location x and time t denoted $u(x, t)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Initially the concentration has the following distribution

$$u(x, 0) = x \quad 0 \leq x \leq \pi$$

The ends of the pipe are closed, so the chemical cannot escape. This can be written as

$$u_x(0, t) = 0 \quad u_x(\pi, t) = 0 \quad t \geq 0$$

(a) Assume that $u(x, t) = X(x)T(t)$ and find ODEs satisfied by X and T.
Solution:

$$X(x)T'(t) = X''(x)T(t) \implies \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Since the left-hand side depends only on t and the left depends only on x, both have to equal a constant, λ:

$$X''(x) - \lambda X(x) = 0$$

$$T'(t) - \lambda T(t) = 0$$

(b) Use the boundary conditions for u to derive boundary conditions for $X(x)$.
Solution:

$$u_x(0, t) = u_x(\pi, t) = 0$$

so we have

$$X'(0)T(t) = 0 \quad X'(\pi)T(t) = 0$$

If $T(t) = 0$ everywhere, the solution would be trivial, so we assume this is not the case. Thus it must be that

$$X'(0) = X'(\pi) = 0.$$

(c) Solve the resulting eigenvalue problem for $X(x)$.
Solution: Let's break down the problem into three different cases.

$\lambda > 0$:
In this case, the general solution is

$$X(x) = C_1 \cosh(\sqrt{\lambda}x) + C_2 \sinh(\sqrt{\lambda}x),$$

Plugging in the initial conditions in this case simply gives $C_1 = C_2 = 0$.

$\lambda = 0$:
In this case, we get $X(x) = C_1 x + C_2$. Plugging in the initial conditions we get $C_1 = 0$ and arbitrary C_2 (but we can take $C_2 = 1$). So the eigenvalue is $\lambda_0 = 0$ and the eigenfunction is $X_0 = 1$.

$\lambda < 0$:
Let $\lambda = -\mu^2$ and we get

$$X(x) = C_1 \cos(\mu x) + C_2 \sin(\mu x).$$

Plugging in the initial conditions we get $C_2 = 0$ and $\sin(\mu\pi) = 0$. Thus $\mu = n$ i.e. the eigenvalues are $\lambda_n = -n^2$ and the corresponding eigenfunctions are $X_n = \cos(nx)$.

(d) For each eigenvalue you found, solve the corresponding ODE for T.
Solution: For $\lambda_0 = 0$, the ODE for T_0 is given by

$$T_0'(t) = 0$$

We denote the solution by T_0 since it corresponds to λ_0. We thus have $T_0 = C_0$ for some constant C_0.
For $\lambda_n = -n^2$, the ODE is $T_n' = -n^2 T_n$ so $T_n = C_n e^{-n^2 t}$, for some constant C_n.

(e) Take linear combinations of all the fundamental solutions $u_n(x,t)$ to get the general solution $u(x,t)$ of this heat equation.
Solution: For λ_0, $u_0(x,t) = C_0/2$ (the factor of half is optional, but it simplifies a calculation in the next part). For λ_n, $u_n(x,t) = C_n \cos(nx) e^{-n^2 t}$ and

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) e^{-n^2 t}.$$

(f) Finally, use the initial condition to find the coefficients C_n.
Solution: Plug in $t = 0$ into the general solution:

$$u(x,0) = x = \frac{C_0}{2} + \sum_{n=1}^{\infty} C_n \cos(nx) \quad x \in [0,\pi]$$

This is just the cosine series expansion for $f(x) = x$. We need to evenly extend $f(x)$ on $[-\pi,\pi]$, so we define $f(x) = -x$ for $x \in [-\pi, 0)$. In other words, we take $f(x) = |x|$ on $[-\pi,\pi]$. We can now compute the coefficients:

$$C_0 = \frac{2}{\pi} \int_0^{\pi} x dx = \pi$$

$$C_n = \frac{2}{\pi}\int_0^\pi x\cos(nx)dx = \frac{2}{n^2\pi}[(-1)^n - 1].$$

Thus the final answer is

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}[(-1)^n - 1]\cos(nx)e^{-n^2 t}$$

2. D'Alembert's formula: For the wave equation $a^2 u_{xx} = u_{tt}$, it turns out that solutions can be written as

$$u(x,t) = F(x+at) + G(x-at)$$

for some functions F and G. This question will guide you through the process of using this formula to solve wave equation problems.

(a) Show that $u(x,t) = F(x+at) + G(x-at)$ satisfies the wave equation.

Solution: Computing the derivatives (using chain rule):

$$u_{xx} = F''(x+at) + G''(x-at)$$

$$u_{tt} = a^2 F''(x+at) + a^2 G(x-at)$$

So we see that

$$a^2 u_{xx} = a^2 F''(x+at) + a^2 G(x-at) = u_{tt}.$$

(b) Suppose we have the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Then show that

$$F(x) + G(x) = f(x)$$

$$a(F'(x) - G'(x)) = 0$$

Solution: Plugging in $t = 0$ into $u(x,t) = F(x+at) + G(x-at)$, we get

$$u(x,0) = F(x) + G(x).$$

From the initial condition, we know $u(x,0) = f(x)$, so this gives us the firs formula.

Similarly, $u_t(x,t) = aF'(x+at) - aG'(x-at)$ thus, $u_t(x,0) = a(F'(x) - G'(x))$. Thus from the second initial condition we get $a(F'(x) - G'(x)) = 0$.

(c) Use the equations from above to show that

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)]$$

solves the wave equation with the given initial conditions.

Solution: From the previous part we got

$$F(x) + G(x) = f(x) \qquad a(F'(x) - G'(x)) = 0$$

Rearranging the second equation we get

$$F'(x) = G'(x)$$

and integrating, we get

$$F(x) = G(x) + C$$

where C is an integrating constant. Plugging this into the first equation,

$$\begin{aligned} F(x) + G(x) &= f(x) \\ (G(x) + C) + G(x) &= f(x) \\ 2G(x) &= f(x) - C \\ G(x) &= \frac{f(x) - C}{2}, \end{aligned}$$

which we can then use to find

$$F(x) = G(x) + C = \frac{f(x) - C}{2} + C = \frac{f(x) + C}{2}.$$

Now we go back to the equation $u(x,t) = F(x+at) + G(x-at)$. Simply use the formulas for F and G in terms of f that we just derived an we get:

$$u(x,t) = \frac{f(x+at) + C}{2} + \frac{f(x-at) - C}{2}$$

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2}$$

3. Neumann problem for Laplace's equation on the disk

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 && 0 \le r \le a \quad 0 \le \theta < 2\pi \\ &u_r(a,\theta) = f(\theta) && 0 \le \theta < 2\pi \end{aligned}$$

Notice that we have prescribed u_r, the derivative of u in the radial direction at the boundary of the disk, instead of u itself. This kind of a problem is known as a *Neumann* problem.

Using the method of separation of variables, find the solution to this problem.

Solution: Assume that

$$u(r,\theta) = R(r)\Theta(\theta)$$

and plug it into the equation, divide by $R\Theta$ and multiply by r^2:

$$r^2\frac{R''}{R} + r\frac{R'}{R} = \frac{-\Theta''}{\Theta}$$

Then from the separation of variables we have

$$\Theta'' + \lambda\Theta = 0$$

$$r^2 R'' + rR - \lambda R = 0$$

Since the θ variable is periodic, for the equation to be well-defined, we must make sure that $\Theta(\theta) = \Theta(\theta + 2\pi)$. This gives us a 2-point boundary value problem in Θ:

$\lambda < 0$:

$$\Theta(\theta) = c_1 e^{\sqrt{\lambda}\theta} + c_2 e^{-\sqrt{\lambda}\theta}$$

Using the periodicity condition,

$$c_1 e^{\sqrt{\lambda}\theta} + c_2 e^{-\sqrt{\lambda}\theta} = c_1 e^{\sqrt{\lambda}\theta} e^{2\pi\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}\theta} e^{-2\pi\sqrt{\lambda}}$$

matching the terms, we get

$$c_1 = c_1 e^{2\pi\sqrt{\lambda}} \quad c_2 = c_2 e^{-2\pi\sqrt{\lambda}}$$

which gives $c_1 = c_2$ (trivial).

$\lambda = 0$:

$$\Theta_0(\theta) = c_1 + c_2\theta$$

For this to be periodic, c_2 must be zero but c_1 can be any constant. Thus $\lambda = 0$ gives non-trivial solutions to the Θ equation. We now plug in $\lambda = 0$ into the equation R to get

$$r^2 R_0'' + rR_0' = 0$$

Taking $w = R_0'$, we get $rw' = -w$. Integrating gives $\ln w = -\ln r + C$, or $w = c_1\frac{1}{r}$. To get R_0 we need to integrate again, and this gives us $R_0(r) = c_1 + c_2 \ln r$. However, at $r = 0$, $ln(r)$ goes to $-\infty$ which does not make sense so we need $c_2 = 0$. Thus we have $R_0(r) = c_1$, and we $u_0(r, \theta) = R_0(r)\Theta_0(\theta) = c_0/2$ (where the 0 denotes the $\lambda = 0$ eigenvalue). We also divide by 2 because this helps simplify a later calculation.

$\lambda > 0$:

$$\Theta(\theta) = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta).$$

For periodicity, we will need $\lambda = n^2$, thus the eigenfunctions

$$\Theta_n(\theta) = A\cos(n\theta) + B\sin(n\theta) \quad n = 1, 2, 3, ...$$

Now we want to solve for R:

$$r^2 R'' + rR' - n^2 R = 0$$

We look for solutions of the form $_m = r^m$. Plugging in, we get $m = \pm n$, thus,

$$R_n(r) = Ar^n + Br^{-n}$$

However, for this to be well defined at $r = 0$, we need $B = 0$

$$R_n(r) = Ar^n$$

Thus we have the general solution:

$$u(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

To use the boundary condition, we differentiate in r:

$$u_r(r, \theta) = \sum_{n=1}^{\infty} nr^{n-1} [A_n \cos(n\theta) + B_n \sin(n\theta)]$$

Plugging in $r = a$ and setting it equal to $f(\theta)$ we get the following equations for A_n and B_n:

$$na^{n-1}A_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta) d\theta$$

$$na^{n-1}B_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta) d\theta$$