§10.6,10.7

Fall MATH 2930

Name: ____________________

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Review

Heat Equation (contd.)

- Non-homogeneous boundary conditions:

$$u(0,t) = T_1 \quad u(L,t) = T_2$$

General solution:

$$u(x,t) = (T_2 - T_1)\frac{x}{L} + T_1 + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right).$$

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - (T_2 - T_1)\frac{x}{L} - T_1\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Insulated ends:

$$u_x(0,t) = 0 \quad u_x(L,t) = 0$$

General solution:

$$u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2\alpha^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right) \qquad c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

Wave Equation

- The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

- Boundary conditions (fixed ends):

$$u(0,t) = 0 \quad u(L,t) = 0 \quad \text{for } t \geq 0$$

- Non-zero initial displacement but zero initial velocity:

$$u(x,0) = f(x) \quad u_t(x,0) = 0 \quad \text{for } 0 < x < L$$

General solution:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi a t}{L}\right) \qquad c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

- Zero initial displacement but non-zero initial velocity:

$$u(x,0) = 0 \quad u_t(x,0) = g(x) \quad \text{for } 0 < x < L$$

$$u(x,t) = \sum_{n}^{\infty} k_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right) \qquad \frac{n\pi a}{L}k_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Laplace Equation

- The 2D Laplace's equation is given in rectangular (Cartesian) coordinates by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and in polar coordinates by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- Dirichlet problem on a rectangular region: $0 < x < a$ and $0 < y < b$ with the boundary conditions

$$\begin{aligned} &u(x,0) = 0, \quad u(x,b) = 0 && 0 < x < a \\ &u(0,y) = 0, \quad u(a,y) = f(y) && 0 < y < b \end{aligned}$$

General solution:

$$u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi x}{b}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad c_n \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b}\int_0^b f(y)\sin\left(\frac{n\pi y}{b}\right)dy$$

- Dirichlet problem on a disk: $r < a$ and $0 \le \theta < 2\pi$ with the boundary condition

$$u(a,\theta) = f(\theta) \quad 0 \le \theta < 2\pi.$$

where f is periodic i.e. $f(0) = f(2\pi)$ (this vaguely acts like a boundary condition in the θ variable).

General solution:

$$u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^n (c_n \cos(n\theta) + k_n \sin(n\theta))$$

$$c_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)d\theta \quad n = 0, 1, 2, ...$$

$$k_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)d\theta \quad n = 1, 2, ...$$

Practice Problems

1. **Heat equation with insulated ends:**
Consider a thin pipe placed along the x-axis with ends at $x = 0$ and $x = \pi$. The pipe is filled with water and a small amount of a certain chemical. The chemical spreads (diffuses) through the pipe and the concentration of the chemical at location x and time t denoted $u(x, t)$ satisfies the equation
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$
Initially the concentration has the following distribution
$$u(x, 0) = x \quad 0 \leq x \leq \pi$$
The ends of the pipe are closed, so the chemical cannot escape. This can be written as
$$u_x(0, t) = 0 \quad u_x(\pi, t) = 0 \quad t \geq 0$$
   (a) Assume that $u(x, t) = X(x)T(t)$ and find ODEs satisfied by X and T.

   (b) Use the boundary conditions for u to derive boundary conditions for $X(x)$.

(c) Solve the resulting eigenvalue problem for $X(x)$.

(d) For each eigenvalue you found, solve the corresponding ODE for T.

(e) Take linear combinations of all the fundamental solutions $u_n(x,t)$ to get the general solution $u(x,t)$ of this heat equation.

(f) Finally, use the initial condition to find the coefficients C_n.

2. D'Alembert's formula: For the wave equation $a^2 u_{xx} = u_{tt}$, it turns out that solutions can be written as

$$u(x,t) = F(x+at) + G(x-at)$$

for some functions F and G. This question will guide you through the process of using this formula to solve wave equation problems.

(a) Show that $u(x,t) = F(x+at) + G(x-at)$ satisfies the wave equation.

(b) Suppose we have the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$. Then show that

$$F(x) + G(x) = f(x)$$
$$a(F'(x) - G'(x)) = 0$$

(c) Use the equations from above to show that

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)]$$

solves the wave equation with the given initial conditions.

3. Neumann problem for Laplace's equation on the disk

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 && 0 \le r \le a \quad 0 \le \theta < 2\pi \\ &u_r(a,\theta) = f(\theta) && 0 \le \theta < 2\pi \end{aligned}$$

Notice that we have prescribed u_r, the derivative of u in the radial direction at the boundary of the disk, instead of u itself. This kind of a problem is known as a *Neumann* problem.

Using the method of separation of variables, find the solution to this problem.